Solutions

EXAM II
MATH 142 -CALCULUS II

There are six questions each worth 10 points.
Carefully read the instruction at the top of each page.

The first page is True or False. For each statement provided you will need to decide whether the statement is true (T) or false (F). You may also write I don't know (IDK). You do not need to justify your answer however if you do so correctly I will award bonus points. A correct answer will earn 2 points, an answer of IDK will automatically earn 1 point, and an incorrect answer will earn no points. You may earn bonus points by writing which theorem (or test) the statement comes from (if it is true) or by providing a counter example (if it is false).

Pay attention to the details:
Example Statement: For all real numbers x: $x^2 \geq x$. Observe that this statement is false since $\left(\frac{1}{2}\right)^2 \ngeq \frac{1}{2}$. Here an answer of F would get 2 points, an answer of IDK would get 1 point and an answer of T would get 0 points. Providing a counter-example (as I did above) will earn you bonus points.

Good luck!

Question 1

Read the instructions on the front page.

- Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ such that $a_n = f(n)$ for each $n \geq 1$. Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^\infty f(x)dx$ is convergent.

T by the integral test.

- Suppose $\sum a_n$ and $\sum b_n$ are series such that $a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

T by the Direct Comparison Test

- If a is a real number then $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

False, if $a=1$ then $\sum_{n=0}^{\infty} a^n$ diverges.

- An alternating series converges if and only if it converges absolutely.

False, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but not absolutely.

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 0$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

False, if $a_n = \frac{1}{n^2}$ then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{1/n^2} = 1$

but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Question 2

a. Determine if the sequence defined by $a_n = \frac{3-5n^3}{n^3+3n^2}$ is convergent. If so, what does it converge to.

b. Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}+5}$ is convergent, absolutely convergent or divergent.

(a) $\lim_{n\to\infty} \frac{3-5n^3}{n^3+3n^2} = \frac{-5}{1} = -5$ so it converges.

(b) Absolutely Convergent. Notice that $n^{3/2}+5 > n^{3/2}$ and thus $\frac{1}{n^{3/2}+5} < \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges as $3/2 > 1$. Thus $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^{3/2}+5}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}+5}$ converges.

Question 3

Find a representation of the repeating decimal $1.\overline{13} = 1.131313\ldots$ as a fraction of two integers.

$$1.131313\ldots = 1 + \frac{13}{100} + \frac{13}{(100)^2} + \frac{13}{(100)^3} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{13}{(100)^n} = 1 + \frac{13}{100} \sum_{n=0}^{\infty} \frac{1}{100^n}$$

$$= 1 + \frac{13}{100}\left(\frac{1}{1-\frac{1}{100}}\right)$$

$$= 1 + \frac{13}{100}\left(\frac{100}{99}\right)$$

$$= 1 + \frac{13}{99} = \frac{112}{99}.$$

Question 4

Does the series $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ converge? Justify your answer.

By the ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)!/10^{(n+1)}}{n!/10^n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{10^n}{10^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n}{10} \to \infty.$$

So the series diverges.

Question 5

Find an exact answer for the following series: (pay attention to the details)

$$\sum_{n=2}^{\infty} \frac{1}{3^n}$$

$$\sum_{n=2}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^{n+2}} = \sum_{n=0}^{\infty} \frac{1}{3^2 \cdot 3^n}$$

$$= \frac{1}{9} \sum_{n=0}^{\infty} \frac{1}{3^n}$$

$$= \frac{1}{9}\left(\frac{1}{1-\frac{1}{3}}\right)$$

$$= \frac{1}{9} \cdot \frac{3}{2} = \frac{1}{6}.$$

Question 6

Determine if the following series are convergent, absolutely convergent or divergent.

a.

$$\sum_{n=1}^{\infty} n!(-e)^{-n}.$$

b.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

(a) $\sum_{n=1}^{\infty} n!(-e)^{-n} = \sum_{n=1}^{\infty} \frac{(-1)^n n!}{e^n}$. By the Ratio Test,

$$\lim_{n\to\infty} \frac{(n+1)!/e^{n+1}}{n!/e^n} = \lim_{n\to\infty} \frac{(n+1)!}{n!}\cdot\frac{e^n}{e^{n+1}} = \lim_{n\to\infty} \frac{n}{e} \to \infty.$$

Thus the series diverges.

(b) By the Alternating Series Test, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and thus the series converges.

However, the series does not converge absolutely as $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges because $1/2 < 1$.